Lorentz torque on a charged sphere rotating in a dielectric fluid in the presence of a uniform magnetic field

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(Received 31 August 1999)

The Lorentz torque exerted by a uniform magnetic field on a charged sphere rotating steadily in a dielectric fluid is calculated to first order in the charge. For a strongly polar fluid and stick boundary conditions the torque is enhanced significantly with respect to its vacuum value. The modification from the vacuum value depends only on the static dielectric constant of the fluid and on the slip parameter. It is independent of the dielectric response of the sphere and of the shape of the radial charge distribution. There is a nonvanishing Lorentz torque, even when the charge is concentrated in the center of the sphere.

PACS number(s): 41.20.−q, 47.65.+a, 77.90.+k

I. INTRODUCTION

The dynamics of polar liquids in interaction with the electromagnetic field is of theoretical as well as technological interest [1,2]. To advance understanding of these complicated systems it is useful to consider simple model situations that can be analyzed theoretically in full detail. A simple situation of interest is the motion of a charged sphere in a dielectric fluid. For simplicity, the fluid may be assumed to be incompressible and viscous.

A charged sphere moving in a dielectric fluid experiences dielectric friction, aside from the usual Stokes friction, due to the relaxation of polarization. If in addition an external magnetic field is applied, then interesting coupling effects arise. In a theory of the Hall effect in dielectric fluids, we have found earlier that the Lorentz force is strongly reduced by the presence of the dielectric fluid [3]. In the following we show that, on the contrary, the Lorentz torque on a rotating charged sphere is significantly enhanced.

First, we recall known theoretical results on dielectric friction and the Hall effect in polar liquids. Thus we consider a nonmagnetic sphere of radius a, charge Q, immersed in an incompressible polar fluid of infinite extent and in the presence of an applied uniform magnetic field $\boldsymbol{B}_0$. The dielectric response of the fluid is characterized by a frequency-dependent dielectric constant $\varepsilon(\omega)$. If the sphere moves steadily with translational velocity $\boldsymbol{U}$, then the force exerted on it is given by

$$\boldsymbol{\mathcal{F}}=-\zeta_t\boldsymbol{U}+h_t\frac{Q}{c}\boldsymbol{U}\times\boldsymbol{B}_0, \tag{1.1}$$

where ζ_t is the translational friction coefficient, h_t is the Hall number, and c is the velocity of light in vacuum. The coefficients ζ_t and h_t depend on the hydrodynamic boundary conditions applied at the surface of the sphere. The friction coefficient ζ_t has been calculated [4] as a function of the charge Q. To second order in the charge it is given by [5]

$$\zeta_t=6\pi\eta a(1-\xi)+C_t\frac{\hat{\varepsilon}_0}{\varepsilon_0^2}\frac{Q^2}{a^3}+O(Q^4), \tag{1.2}$$

where the first term is the usual Stokes contribution for a neutral sphere. Here η is the shear viscosity of the fluid, and ξ is the slip parameter taking the value $\xi=0$ for stick and $\xi=\frac{1}{3}$ for perfect slip boundary condition. The coefficient $\varepsilon_0=\varepsilon(0)$ is the value of the dielectric constant at zero frequency, and the coefficient $\hat{\varepsilon}_0$ is defined by

$$\hat{\varepsilon}_0=\lim_{\omega\to 0}\frac{\varepsilon(\omega)-\varepsilon(0)}{i\omega}. \tag{1.3}$$

The coefficient C_t was found to be [5]

$$C_t=\tfrac{1}{280}[17-54\xi+177\xi^2]. \tag{1.4}$$

This value agrees with an earlier result of Hubbard and Onsager [6] for perfect stick ($\xi=0$) and perfect slip ($\xi=\frac{1}{3}$), but differs for intermediate values $0<\xi<\frac{1}{3}$ of the slip parameter due to the use of a different boundary condition. One of us has argued that the discontinuity of the electrostatic stress tensor must be accounted for in the boundary condition [5]. It was shown later that the modified boundary condition is consistent with Onsager symmetry [7].

The Hall number h_t in Eq. (1.1) has been evaluated to zeroth order in the charge [9]. We found $h_t=h_t^{(0)}+O(Q^2)$ with

$$h_t^{(0)}=1-C(\xi)\frac{\varepsilon_0-1}{\varepsilon_0} \tag{1.5}$$

with coefficient

$$C(\xi)=\tfrac{7}{16}-\tfrac{3}{16}\xi^2. \tag{1.6}$$

This shows a strong reduction of the Lorentz force from its vacuum value in a strongly polarizable fluid with $\varepsilon_0\gg 1$. We have compared with earlier theoretical results [8]. The Hall effect in electrolyte solutions has been measured by Gérard *et al.* [9].

In the following, we consider a sphere rotating steadily with angular velocity $\mathbf{\Omega}$. The torque exerted on the sphere is given by

$$\boldsymbol{\mathcal{T}}=-\zeta_r\mathbf{\Omega}+h_r\boldsymbol{\mathcal{T}}_{La}, \tag{1.7}$$

where ζ_r is the rotational friction coefficient, h_r is the rotational Hall number, and $\boldsymbol{\mathcal{T}}_{La}$ is the Lorentz torque exerted by the magnetic field if the sphere were in vacuum with the charge concentrated on its surface,

$$\boldsymbol{\mathcal{T}}_{La}=\frac{1}{3c}Qa^2\mathbf{\Omega}\times\boldsymbol{B}_0. \tag{1.8}$$

The rotational friction coefficient ζ_r has been calculated as a function of the charge [10]. To second order in the charge it is given by

$$\zeta_r=8\pi\eta a^3(1-3\xi)+C_r\frac{\hat{\varepsilon}_0}{\varepsilon_0^2}\frac{Q^2}{a}+O(Q^4), \tag{1.9}$$

where the first term is the usual Stokes contribution for a neutral sphere. The coefficient C_r was found to be [10]

$$C_r=\tfrac{3}{14}(1-3\xi)^2. \tag{1.10}$$

In the following, we calculate the rotational Hall number $h_r=h_r^{(0)}+O(Q^2)$ to lowest order in the charge. We find that for a strongly polarizable fluid with $\varepsilon_0\gg 1$ the Lorentz torque is enhanced by about 50% from its vacuum value if the charge is concentrated on the surface of the sphere. The calculation follows similar lines as that for the translational Hall number $h_t^{(0)}$.

II. BASIC EQUATIONS

We have shown in Ref. [11] on the basis of the de Groot–Mazur equations [12,13] that in a steady-state situation the total momentum balance equation for fluid and electromagnetic fields is given by

$$\nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v})-\nabla\cdot(\boldsymbol{\sigma}^S_{hyd}+\boldsymbol{\sigma}^S_{em})=0, \tag{2.1}$$

where ρ is the mass density, $\boldsymbol{v}(\boldsymbol{r})$ is the flow velocity, and $\boldsymbol{\sigma}^S_{hyd}$ and $\boldsymbol{\sigma}^S_{em}$ are the symmetric parts of the hydrodynamic and electromagnetic stress tensor. The symmetrized hydrodynamic stress tensor is given by

$$\sigma^S_{hyd,\alpha\beta}=\eta(\partial_\alpha v_\beta+\partial_\beta v_\alpha)-p\,\delta_{\alpha\beta}, \tag{2.2}$$

where p is the pressure. The symmetrized electromagnetic stress tensor is

$$\boldsymbol{\sigma}^S_{em}=\frac{1}{8\pi}[\boldsymbol{DE}+\boldsymbol{ED}+\boldsymbol{BH}+\boldsymbol{HB}-(\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{H})\mathbf{1}]+\frac{1}{2c}\boldsymbol{v}(\boldsymbol{P}\times\boldsymbol{B})+\frac{1}{2c}(\boldsymbol{P}\times\boldsymbol{B})\boldsymbol{v}. \tag{2.3}$$

The dielectric displacement $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$ and the polarization $\boldsymbol{P}$ by

$$\boldsymbol{D}=\boldsymbol{E}+4\pi\boldsymbol{P}. \tag{2.4}$$

The magnetic induction $\boldsymbol{B}$ is related to the magnetic field $\boldsymbol{H}$ and the magnetization $\boldsymbol{M}$ by

$$\boldsymbol{B}=\boldsymbol{H}+4\pi\boldsymbol{M}. \tag{2.5}$$

In the derivation of Eq. (2.3), crucial use is made of irreversible thermodynamics [12]. The last two terms on the right-hand side are necessary to guarantee a positive entropy production and a properly Galilei invariant torque density [13]. Intrinsic rotation of the fluid molecules has been taken into account. In the steady-state limit, considered here, the average intrinsic angular momentum has relaxed to a constant value determined by the local torque density and the rotational viscosity of the fluid. As a consequence, the antisymmetric parts of the hydrodynamic and electromagnetic stress tensor cancel, and only the symmetric parts remain in Eq. (2.1). Other proposals made for the electromagnetic stress tensor violate Galilei invariance of the torque density, and have not been shown to guarantee a positive entropy production in combination with phenomenological relaxation equations. Therefore we consider in the following only the form Eq. (2.3). For further details we refer the reader to Ref. [13].

In the fluid the fields satisfy the static Maxwell equations

$$\begin{aligned}\nabla\cdot\boldsymbol{D}=0,\quad \nabla\cdot\boldsymbol{B}=0,\\ \nabla\times\boldsymbol{E}=\mathbf{0},\quad \nabla\times\boldsymbol{H}=\mathbf{0}.\end{aligned} \tag{2.6}$$

These equations must be supplemented with constitutive equations for polarization and magnetization. The magnetization is given by [14]

$$\boldsymbol{M}=-\frac{1}{c}\boldsymbol{v}\times\boldsymbol{P}. \tag{2.7}$$

We have omitted terms of the form $(\boldsymbol{v}/c)\boldsymbol{M}\times\boldsymbol{E}$ from Eq. (2.3) since these are of order v^2/c^2. The polarization is given by

$$\boldsymbol{P}=\kappa_0\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)+\hat{\boldsymbol{d}}^f, \tag{2.8}$$

where $\kappa_0=\kappa(0)$ is the zero-frequency susceptibility, which is related to the dielectric constant by $\varepsilon(\omega)=1+4\pi\kappa(\omega)$. The last term in Eq. (2.8) is the additional polarization due to transport by the flow. The detailed expression up to terms linear in $\mathbf{\Omega}$ will be given below.

We take the sphere to be centered at the origin. We regard the steady rotational velocity $\mathbf{\Omega}$ as a small quantity and linearize the equations in terms of it. To lowest order, both the sphere and the fluid are at rest. In this situation the dielectric displacement $\boldsymbol{D}_0$ and the electric field $\boldsymbol{E}_0$ in the fluid are given by

$$\boldsymbol{D}_0=Q\frac{\hat{\boldsymbol{r}}}{r^2},\quad \boldsymbol{E}_0=\frac{Q}{\varepsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2}\quad (r>a), \tag{2.9}$$

and the polarization $\boldsymbol{P}_0$ is given by

$$\boldsymbol{P}_0=\frac{\kappa_0}{\varepsilon_0}Q\frac{\hat{\boldsymbol{r}}}{r^2}\quad (r>a). \tag{2.10}$$

The zeroth-order magnetic field is $\boldsymbol{H}_0=\boldsymbol{B}_0$. The unperturbed fluid velocity $\boldsymbol{v}_0$ and the additional polarization $\hat{\boldsymbol{d}}_0^f$ vanish identically.

Quantities linear in $\boldsymbol{\Omega}$ are denoted by a subscript 1. Since $\boldsymbol{v}_0=0$, the linearized momentum balance equation (2.1) becomes

$$\eta\nabla^2\boldsymbol{v}_1-\nabla p_1=-\boldsymbol{G}_1, \tag{2.11}$$

where $\boldsymbol{G}_1$ is defined by

$$\boldsymbol{G}_1=\nabla\cdot\boldsymbol{\sigma}^S_{em,1}. \tag{2.12}$$

We may formally regard $\boldsymbol{G}_1(\boldsymbol{r})$ as a force density acting on the fluid. The pressure p_1 follows from the condition of incompressibility $\nabla\cdot\boldsymbol{v}_1=0$. The first-order fields $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, and $\boldsymbol{H}_1$ satisfy the Maxwell equations (2.6). The first-order magnetization is given by

$$\boldsymbol{M}_1=-\frac{1}{c}\boldsymbol{v}_1\times\boldsymbol{P}_0. \tag{2.13}$$

The first-order polarization is given by

$$\boldsymbol{P}_1=\kappa_0\left(\boldsymbol{E}_1+\frac{1}{c}\boldsymbol{v}_1\times\boldsymbol{B}_0\right)+\hat{\boldsymbol{d}}_1^f, \tag{2.14}$$

with the additional polarization [11]

$$\hat{\boldsymbol{d}}_1^f=-\frac{\hat{\varepsilon}_0}{4\pi}[(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{E}_0-\tfrac{1}{2}(\nabla\times\boldsymbol{v}_1)\times\boldsymbol{E}_0]. \tag{2.15}$$

We have shown in Ref. [14] that, when terms of order $1/c^2$ are neglected and use is made of the uniformity of $\boldsymbol{B}_0$ and the condition of incompressibility, the expression for the force density is found to be

$$\boldsymbol{G}_1=-\tfrac{1}{2}\boldsymbol{E}_0(\nabla\cdot\hat{\boldsymbol{d}}_1^f)-\tfrac{1}{2}\boldsymbol{E}_0\times(\nabla\times\hat{\boldsymbol{d}}_1^f)+\frac{\kappa_0}{c}[(\nabla\times\boldsymbol{v}_1)\times(\boldsymbol{E}_0\times\boldsymbol{B}_0)-\boldsymbol{B}_0\times(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{E}_0]. \tag{2.16}$$

The fluid equation of motion (2.11) must be supplemented with a boundary condition for the fluid velocity at the surface of the sphere. Since the fluid cannot penetrate the sphere, the radial component of the fluid velocity v_{1r} must vanish at $r=a$. A natural generalization of the usual tangential condition is [4]

$$\boldsymbol{v}_{1t}=\frac{\lambda}{\eta}[\hat{\boldsymbol{r}}\cdot(\boldsymbol{\sigma}^S_{hyd,1}+\boldsymbol{\sigma}^S_{em,1})_t], \tag{2.17}$$

where λ is a proportionality constant taking the value 0 for stick and ∞ for perfect slip. The slip parameter ξ in Eq. (1.2) is related to λ by $\xi=\lambda/(a+3\lambda)$. The square brackets in Eq. (2.17) indicate the jump at the surface $r=a$. The solution of Eq. (2.11) with the above boundary conditions is equivalent to the solution of

$$\eta\nabla^2\boldsymbol{v}_1-\nabla p_1=-\boldsymbol{F}_1(\boldsymbol{r}), \tag{2.18}$$

with the force density

$$\boldsymbol{F}_1(\boldsymbol{r})=\boldsymbol{G}_1(\boldsymbol{r})\theta(r-a)+\boldsymbol{g}_1(\hat{\boldsymbol{r}})\delta(r-a+) \tag{2.19}$$

and the usual hydrodynamic boundary conditions, provided the surface force density $\boldsymbol{g}_1(\hat{r})$ is evaluated from

$$\boldsymbol{g}_1(\hat{\boldsymbol{r}})=[\hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}^S_{em,1}]. \tag{2.20}$$

The surface force density was missing in the theory of Hubbard and Onsager [6].

Finally, we must specify the electrical properties of the sphere. For conceptual clarity we assume that the charge density and the polarization vanish identically in the shell $a-\delta<r<a$, where δ may be infinitesimal. In this shell the electromagnetic stress tensor takes the vacuum form and is automatically symmetric. We assume that the sphere has quadrupolar polarizability α_2.

III. TORQUE EXERTED ON THE SPHERE

The force and torque exerted on the sphere may be evaluated from integrals of the stress tensor over a spherical surface S_{a+} just enclosing it. To first order in $\boldsymbol{\Omega}$ the force is given by

$$\mathcal{F}_1=\int_{S_{a+}}(\boldsymbol{\sigma}^S_{hyd,1}+\boldsymbol{\sigma}^S_{em,1})\cdot\hat{\boldsymbol{r}}\,dS. \tag{3.1}$$

Using the momentum-balance equation for the fluid and Gauss' theorem, we may rewrite Eq. (3.1) in the form

$$\mathcal{F}_1=\int_{S_\infty}(\boldsymbol{\sigma}^S_{hyd,1}+\boldsymbol{\sigma}^S_{em,1})\cdot\hat{\boldsymbol{r}}\,dS, \tag{3.2}$$

where the integral is over a spherical surface of arbitrarily large radius. By spherical symmetry the force vanishes identically. Similarly, the first-order torque is

$$\mathcal{T}_1=\int_{S_{a+}}\boldsymbol{r}\times[(\boldsymbol{\sigma}^S_{hyd,1}+\boldsymbol{\sigma}^S_{em,1})\cdot\hat{\boldsymbol{r}}]dS. \tag{3.3}$$

This may be transformed to

$$\mathcal{T}_1=\int_{S_\infty}\boldsymbol{r}\times[(\boldsymbol{\sigma}^S_{hyd,1}+\boldsymbol{\sigma}^S_{em,1})\cdot\hat{\boldsymbol{r}}]dS. \tag{3.4}$$

The equations presented in the preceding section completely determine the problem. We shall not attempt to solve the equations exactly, but resort to a perturbation expansion in powers of the charge Q. We show by explicit calculation that to terms linear in the charge the force $\mathcal{F}_1$ does indeed vanish. The calculation of $\mathcal{T}_1$ to terms linear in Q yields the rotational Hall number $h_1^{(0)}$.

The first few terms in our perturbation expansion are

$$\boldsymbol{v}_1=\boldsymbol{v}_1^{(0)}+\boldsymbol{v}_1^{(1)}+\cdots,$$
$$\boldsymbol{E}_1=\boldsymbol{E}_1^{(0)}+\boldsymbol{E}_1^{(1)}+\cdots, \tag{3.5}$$
$$\boldsymbol{B}_1=\boldsymbol{B}_1^{(1)}+\cdots,$$

where the superscript indicates the power of the charge Q. It follows from Eqs. (2.16) and (2.20) that the force density $\boldsymbol{G}_1$ and the surface force density $\boldsymbol{g}_1$ start with terms $\boldsymbol{G}_1^{(1)}$ and $\boldsymbol{g}_1^{(1)}$ of first order in the charge. Hence to zeroth order the velocity field $\boldsymbol{v}_1^{(0)}$ satisfies the homogeneous linear Navier-Stokes equations. These have the well-known solution

$$\boldsymbol{v}_1^{(0)}(\boldsymbol{r})=A_T\frac{\boldsymbol{\Omega}\times\hat{\boldsymbol{r}}}{r^2} \quad (r>a), \tag{3.6}$$

where A_T is a hydrodynamic scattering coefficient with the value

$$A_T=(1-3\xi)a^3. \tag{3.7}$$

It follows from Eq. (2.16) that the first-order force density $\boldsymbol{G}_1^{(1)}$ is given by

$$\boldsymbol{G}_1^{(1)}=\frac{\kappa_0 Q}{2\epsilon_0 c}\frac{A_T}{r^5}[6\hat{\boldsymbol{r}}\boldsymbol{\Omega}\cdot(\mathbf{1}-\hat{\boldsymbol{r}}\hat{\boldsymbol{r}})\cdot\boldsymbol{B}_0-2\boldsymbol{\Omega}(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_0)+2\boldsymbol{B}_0(\hat{\boldsymbol{r}}\cdot\boldsymbol{\Omega})]. \tag{3.8}$$

In a volume element moving with the flow the applied magnetic field $\boldsymbol{B}_0$ acts as an electric field polarizing the fluid. The polarization in turn generates an electric field. To zeroth order in the charge this electric field $\boldsymbol{E}_1^{(0)}$ satisfies the equations

$$\begin{aligned}\nabla\cdot\boldsymbol{E}_1^{(0)}&=-\frac{4\pi\kappa_0}{\varepsilon_0 c}\nabla\cdot(\boldsymbol{v}_1^{(0)}\times\boldsymbol{B}_0),\\ \nabla\times\boldsymbol{E}_1^{(0)}&=0 \quad [r>(a-\delta)].\end{aligned} \tag{3.9}$$

Inserting Eq. (3.6), we find

$$\nabla\cdot(\boldsymbol{v}_1^{(0)}\times\boldsymbol{B}_0)=\frac{A_T}{r^3}[3(\hat{\boldsymbol{r}}\cdot\boldsymbol{\Omega})(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_0)-\boldsymbol{\Omega}\cdot\boldsymbol{B}_0]. \tag{3.10}$$

We write $\boldsymbol{E}_1^{(0)}=-\nabla\phi_1^{(0)}$ and make the ansatz

$$\phi_1^{(0)}=g(r)[3(\boldsymbol{r}\cdot\boldsymbol{\Omega})(\boldsymbol{r}\cdot\boldsymbol{B}_0)-r^2\boldsymbol{\Omega}\cdot\boldsymbol{B}_0]. \tag{3.11}$$

Substituting in Eq. (3.9), we find the solution

$$g(r)=-\frac{2\pi\kappa_0}{3\varepsilon_0 c}\frac{A_T}{r^3}+\frac{W}{r^5} \quad (r>a), \tag{3.12}$$

with a coefficient W that must be found by application of the jump conditions at $r=a$. In the neutral shell the radial function must have the form

$$g(r)=Y\left(\frac{\alpha_2}{r^5}-1\right) \quad [(a-\delta)<r<a], \tag{3.13}$$

where α_2 is the quadrupolar polarizability of the sphere, and Y is a second coefficient determined by the jump conditions at $r=a$. We do not need the explicit expressions for W and Y.

The first-order surface force density

$$\boldsymbol{g}_1^{(1)}=[\hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}_{em,1}^{S(1)}] \tag{3.14}$$

can now be calculated as in Sec. VI of [10]. We find

$$\boldsymbol{g}_1^{(1)}=\frac{Q}{4\pi a^2}[\boldsymbol{E}_1^{(0)}(a+)-\boldsymbol{E}_1^{(0)}(a-)]. \tag{3.15}$$

With Eq. (3.11) this becomes

$$\boldsymbol{g}_1^{(1)}=\frac{Q}{4\pi}[g'(a+)-g'(a-)][\boldsymbol{\Omega}\cdot\boldsymbol{B}_0-3(\hat{\boldsymbol{r}}\cdot\boldsymbol{\Omega})(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_0)]\hat{\boldsymbol{r}}. \tag{3.16}$$

Together with the bulk force density $\boldsymbol{G}_1^{(1)}$ in Eq. (3.8) this can be used to calculate the first-order flow.

IV. FIRST-ORDER FLOW

We can now solve for the first-order flow velocity $\boldsymbol{v}_1^{(1)}$ from the linear Navier-Stokes equation (2.18) with inhomogeneous terms given by the volume force density $\boldsymbol{G}_1^{(1)}$ and the surface force density $\boldsymbol{g}_1^{(1)}$. To perform the calculation explicitly it is necessary to expand the force densities in terms of vector spherical harmonics. We have demonstrated the technique in Sec. IX of Ref. [5]. The complete flow pattern can be obtained, but for the calculation of the force and torque on the sphere it suffices to find the asymptotic flow. This is of the form

$$\boldsymbol{v}_1^{(1)}(\boldsymbol{r})=\frac{1}{c}A_1^{(1)}\frac{(\boldsymbol{\Omega}\times\boldsymbol{B}_0)\times\hat{\boldsymbol{r}}}{r^2}+O(r^{-3}). \tag{4.1}$$

with the coefficient $A_1^{(1)}$ to be determined. A flow of the type (4.1) can be generated only by vector spherical harmonics $\boldsymbol{C}_{1m}$ of order $\ell=1$. There are no harmonics $\boldsymbol{A}_{1m}$ and $\boldsymbol{B}_{1m}$ of order $\ell=1$, so that the force on the sphere vanishes. In order to determine the torque on the sphere, we must pick out the amplitudes of the harmonics $\boldsymbol{C}_{1m}$ in the force densities.

Since the flow is linear in $\boldsymbol{\Omega}$, we may consider separately the cases with $\boldsymbol{\Omega}\parallel\boldsymbol{B}_0$ and $\boldsymbol{\Omega}\perp\boldsymbol{B}_0$. We consider first the case $\boldsymbol{\Omega}\perp\boldsymbol{B}_0$ and choose coordinates such that $\boldsymbol{\Omega}=\Omega\boldsymbol{e}_y$ and $\boldsymbol{B}_0=B_0\boldsymbol{e}_z$. Then Eq. (3.8) becomes

$$\boldsymbol{G}_1^{(1)}=\frac{\kappa_0 Q}{2\varepsilon_0 c}\frac{A_T}{r^5}[6n_yn_z\hat{\boldsymbol{r}}+2n_z\boldsymbol{e}_y-2n_y\boldsymbol{e}_z]\Omega B_0 \tag{4.2}$$

and Eq. (3.16) becomes

$$\boldsymbol{g}_1^{(1)}=-3\frac{Q}{4\pi}[g'(a+)-g'(a-)]n_yn_z\hat{\boldsymbol{r}}\Omega B_0. \tag{4.3}$$

We need the following three vector spherical harmonics in Cartesian coordinates [15]

$$\begin{aligned}A_{2\gamma}^{\alpha\beta}&=3\delta_{\alpha\gamma}+3\delta_{\beta\gamma}-2n_\gamma\delta_{\alpha\beta},\\ B_{2\gamma}^{\alpha\beta}&=3\delta_{\alpha\gamma}+3\delta_{\beta\gamma}+3n_\gamma\delta_{\alpha\beta}-15n_\alpha n_\beta n_\gamma,\\ C_{1\gamma}^{\alpha}&=\varepsilon_{\gamma\alpha\beta}n_\beta,\end{aligned} \tag{4.4}$$

where the subscript γ denotes the vector component and the superscripts label the harmonics. It may be seen that

$$n_yn_z\hat{\boldsymbol{r}}=\frac{1}{15}(\boldsymbol{A}_2^{yx}-\boldsymbol{B}_2^{yz}), \tag{4.5}$$

$$n_y\boldsymbol{e}_z=\frac{1}{6}(\boldsymbol{A}_2^{yz}+3\boldsymbol{C}_1^x),$$

$$n_z\boldsymbol{e}_y=\frac{1}{6}(\boldsymbol{A}_2^{yz}-3\boldsymbol{C}_1^x),$$

with $\boldsymbol{C}_1^x=\boldsymbol{e}_x\times\hat{\boldsymbol{r}}$. Now it is evident that only $\boldsymbol{G}_1^{(1)}$ contains a vector spherical harmonic of order unity. A force density $\boldsymbol{C}_1^x\delta(r-s)$ causes a flow velocity

$$\boldsymbol{v}_{c_1^x}(\boldsymbol{r})=\frac{1}{3\eta}\left[r\,\theta(s-r)+\frac{s^3}{r^2}\theta(r-s)-\frac{A_T}{r^2}\right]\boldsymbol{C}_1^x \quad (r>a). \tag{4.6}$$

From Eqs. (4.2) and (4.5) we find the contribution from different values of s.

Next we consider the case $\boldsymbol{\Omega}\|\boldsymbol{B}_0$ and choose coordinates such that $\boldsymbol{\Omega}=\Omega\boldsymbol{e}_z$ and $\boldsymbol{B}_0=B_0\boldsymbol{e}_z$. With the vector spherical harmonic $\hat{\boldsymbol{B}}_0=\hat{\boldsymbol{r}}$ we find

$$n_z^2\hat{\boldsymbol{r}}=\tfrac{1}{15}(\boldsymbol{A}_2^{zz}-\boldsymbol{B}_2^{zz}-5\hat{\boldsymbol{B}}_0). \tag{4.7}$$

(We use a special notation for the spherical harmonic $\hat{\boldsymbol{B}}_0$ to distinguish from the magnetic field.) It is evident from Eqs. (3.8) and (3.16) that for this case the force densities have no contribution from vector spherical harmonics of order unity.

Combining the above results, we find for the coefficient $A_1^{(1)}$ in Eq. (4.1),

$$A_1^{(1)}=\frac{\kappa_0 Q}{\varepsilon_0}\frac{A_T}{3\eta a}\left(1-\frac{A_T}{4a^3}\right). \tag{4.8}$$

The coefficient vanishes for perfect slip, as one would expect, since then there is no zero-order flow.

V. TORQUE ON THE SPHERE

The torque on the sphere is conveniently calculated from Eq. (3.4), since in contrast to Eq. (3.3) this requires knowledge of only the asymptotic fields. We abbreviate Eq. (3.4) as

$$\mathcal{T}_1=\mathcal{T}_{1,1}+\mathcal{T}_{2,1}. \tag{5.1}$$

Here, the first term

$$\mathcal{T}_{1,1}=\int_{S_\infty}\boldsymbol{r}\times(\boldsymbol{\sigma}^S_{hyd,1}\cdot\hat{\boldsymbol{r}})dS \tag{5.2}$$

may be evaluated from the asymptotic flow calculated in Sec. IV. This yields to first order in Q

$$\mathcal{T}_{1,1}^{(1)}=-\frac{8\pi}{c}\eta A_1^{(1)}\boldsymbol{\Omega}\times\boldsymbol{B}_0. \tag{5.3}$$

The second term in Eq. (5.1),

$$\mathcal{T}_{2,1}=\int_{S_\infty}\boldsymbol{r}\times(\boldsymbol{\sigma}^S_{em,1}\cdot\hat{\boldsymbol{r}})dS \tag{5.4}$$

may be evaluated from the contributions to the electromagnetic stress tensor listed in Eq. (3.17) of Ref. [3]. It is easily seen that only the magnetic part of the stress tensor contributes asymptotically, so that we get to first order in the charge

$$\mathcal{T}_{2,1}^{(1)}=\int_{S_\infty}\boldsymbol{r}\times(\boldsymbol{\sigma}^{S(1)}_{mag,1}\cdot\hat{\boldsymbol{r}})dS \tag{5.5}$$

with the magnetic stress tensor

$$\boldsymbol{\sigma}^{S(1)}_{mag,1}=\frac{1}{4\pi}[\boldsymbol{B}_0\boldsymbol{B}_1^{(1)}+\boldsymbol{B}_1^{(1)}\boldsymbol{B}_0-(\boldsymbol{B}_0\cdot\boldsymbol{B}_1^{(1)})\mathbf{1}]. \tag{5.6}$$

In order to evaluate the integral (5.5) it suffices to know the asymptotic behavior of the magnetic induction $\boldsymbol{B}_1^{(1)}$. This is given by

$$\boldsymbol{B}_1^{(1)}(\boldsymbol{r})=\frac{-\mathbf{1}+3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}\cdot\boldsymbol{m}_1^{(1)}+O(r^{-4}), \tag{5.7}$$

where $\mathbf{m}_1^{(1)}$ is the total magnetic moment to first order in Ω and Q. Substituting into Eq. (5.6) we find from Eq. (5.5)

$$\mathcal{T}_{2,1}^{(1)}=\boldsymbol{m}_1^{(1)}\times\boldsymbol{B}_0, \tag{5.8}$$

the usual expression for the torque exerted by a magnetic field on a magnetic moment. The magnetic moment is given by

$$\boldsymbol{m}_1^{(1)}=\frac{1}{2c}\int_{r\leqslant a}\boldsymbol{r}\times(\boldsymbol{\Omega}\times\boldsymbol{r})q(r)d\boldsymbol{r}+\int_{r>a}\boldsymbol{M}_1^{(1)}d\boldsymbol{r}, \tag{5.9}$$

where $q(r)$ is the charge distribution of the sphere and $\boldsymbol{M}_1^{(1)}$ is the magnetization of the fluid following from Eq. (2.13). We write the first term as

$$\frac{1}{2c}\int_{r\leqslant a}\boldsymbol{r}\times(\boldsymbol{\Omega}\times\boldsymbol{r})q(r)d\boldsymbol{r}=\frac{1}{3c}Qa_q^2\boldsymbol{\Omega}. \tag{5.10}$$

If the charge is concentrated on the surface, then $a_q=a$. From Eqs. (2.10), (2.13), and (3.6) one finds for the second term in Eq. (5.9):

$$\int_{r>a}\boldsymbol{M}_1^{(1)}d\boldsymbol{r}=\frac{8\pi}{3}\frac{\kappa_0 Q}{\varepsilon_0 c}\frac{A_T}{a}\boldsymbol{\Omega}. \tag{5.11}$$

Altogether we find for the torque $\mathcal{T}_1$ to first order in the charge

$$\mathcal{T}_1^{(1)}=\frac{1}{3c}Qa^2h_1^{(0)}\boldsymbol{\Omega}\times\boldsymbol{B}_0 \tag{5.12}$$

with rotational Hall number

$$h_1^{(0)}=\frac{a_q^2}{a^2}+\frac{\varepsilon_0-1}{2\varepsilon_0}(1-3\xi)^2. \tag{5.13}$$

For $\varepsilon_0\gg 1$ and the stick boundary condition this shows a significant enhancement with respect to the first term. This is in contrast to translational motion, where the Hall number $h_t^{(0)}$ is decreased from unity by the electromagnetohydrodynamic coupling. Note that the torque $\mathcal{T}_1$ is nonvanishing even if the charge is concentrated in the center of the sphere. For a macroscopic sphere the slip parameter ξ in Eq. (5.13) must be put equal to zero. Remarkably, the second term in Eq. (5.13) then depends only on the static dielectric constant of the fluid.

VI. DISCUSSION

We have shown that the torque on a charged sphere rotating in a dielectric fluid in the presence of a uniform magnetic field is enhanced significantly from its vacuum value if the fluid is strongly polar. In view of the often disputed questions concerning the correct expression for the electromagnetic stress tensor in a polarizable medium [13,16–18], it would be of interest to compare our theoretical prediction with experiment. In our opinion, there is a need for experimental verification of the theoretical expressions for all four transport coefficients $\zeta_t, \zeta_r, h_t, h_r$, considered in the Introduction. The relaxation of polarization of the fluid causes dielectric friction with corresponding dissipative contributions to the friction coefficients ζ_t and ζ_r. The Hall coefficients h_t and h_r are nondissipative. For slow motion they are determined by the static dielectric constant of the fluid.

We suggest experimental study of the slow macroscopic rigid body motion of a charged sphere in a viscous dielectric fluid such as water in the presence and absence of an applied magnetic field. For example, one could study the settling of a charged sphere in gravity in the presence of a horizontal magnetic field. In such a situation, the Lorentz force causes a deviation from the vertical in proportion to the translational Hall coefficient h_t. If in addition the sphere rotates, for example, because it rolled down an incline before falling into the liquid, then the Lorentz torque, calculated above, affects the rotation. The sphere may have a nonconducting insulating shell. We predict an effect on the rotation, even if the charge is located at the center of the sphere.

A numerical estimate of the torque shows that its effect should be detectable. For example, consider a sphere of radius $a=1$ cm, with charge $Q=30$ esu, rotating with angular velocity $\Omega=10^2\,\mathrm{sec}^{-1}$ in a magnetic field $B=10^4$ G. Then the product $Qa^2\Omega B/c$ in Eq. (5.12) takes the value 10^{-3} dyn cm. In water with static dielectric constant $\varepsilon_0=80$ this is the order of magnitude of the torque for any spherical distribution of the charge, since then the rotational Hall number in Eq. (5.13) is of order unity. Although the torque is not large, it should be measurable. A conceivable experimental situation might involve a charged sphere rotating about an axis perpendicular to a strong magnetic field. The torque is perpendicular to both the magnetic field and the axis of rotation or viscous torque, and tends to tilt the axis. The Lorentz torque or the tilt should be measured. Such a measurement would provide a welcome test of the theory.

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